CONVERGENCE OF MULTIPLE ERGODIC AVERAGES ALONG CUBES FOR SEVERAL COMMUTING TRANSFORMATIONS

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Abstract. We prove the norm convergence of multiple ergodic averages along cubes for several commuting transformations, and derive corresponding combinatorial results. The method we use relies primarily on the "magic extension" established recently by B. Host.

1. Introduction

1.1. **Results.** By a system, we mean a probability space endowed with a single or several commuting measure preserving transformations. We prove the following result regarding the convergence of multiple ergodic averages along cubes for several commuting transformations :

Theorem 1.1. *Let $d \geq 1$ be an integer and $(X, \mathcal{B}, \mu, T_1, \cdots, T_d)$ be a system. Let $f_\epsilon, \epsilon \in \{0,1\}^d \setminus \{00\cdots 0\}$ be $2^d - 1$ bounded measurable functions on X. Then the averages*

$$\prod_{i=1}^{d} \frac{1}{N_i - M_i} \sum_{\substack{n_i \in [M_i, N_i) \\ i=1,\dots,d}} \prod_{\substack{\epsilon \in \{0,1\}^d \\ \epsilon \neq 00\cdots 0}} T_1^{n_1\epsilon_1} \cdots T_d^{n_d\epsilon_d} f_\epsilon \tag{1}$$

converge in $L^2(\mu)$ for all sequences of intervals $[M_1, N_1), \dots, [M_d, N_d)$ whose lengths $N_i - M_i$ $(1 \leq i \leq d)$ tend to ∞.

To illustrate, when $d = 2$, the average (1) is

$$\frac{1}{(N_1 - M_1) \times (N_2 - M_2)} \sum_{\substack{n_1 \in [M_1, N_1) \\ n_2 \in [M_2, N_2)}} T_1^{n_1} f_{10} \cdot T_2^{n_2} f_{01} \cdot T_1^{n_1} T_2^{n_2} f_{11}. \tag{2}$$

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When Theorem 1.1 is restricted to the case that each function f_ϵ is the indicator function of a measurable set, we have the following lower bound for these averages:

Theorem 1.2. *Let $(X, \mathcal{B}, \mu, T_1, \cdots, T_d)$ be a system and let $A \in \mathcal{B}$. Then the limit of the averages*

$$\prod_{i=1}^{d} \frac{1}{N_i - M_i} \sum_{\substack{n_i \in [M_i, N_i) \\ i=1,\dots,d}} \mu\Big(\bigcap_{\epsilon \in \{0,1\}^d} T_1^{-n_1\epsilon_1} \cdots T_d^{-n_d\epsilon_d} A\Big) \tag{3}$$

exists and is greater than or equal to $\mu(A)^{2^d}$ for all sequences of intervals $[M_1, N_1), \dots, [M_d, N_d)$ whose lengths $N_i - M_i$ $(1 \le i \le d)$ tend to ∞.

Recall that the upper density $d^*(A)$ of a set $A \subset \mathbb{Z}^d$ is defined to be

$$d^*(A) = \limsup_{\substack{N_i \to \infty \\ 1 \le i \le d}} \prod_{i=1}^{d} \frac{1}{N_i} |A \cap [1, N_1] \times \cdots \times [1, N_d]|.$$

A subset E of $\mathbb{Z}^d$ is said to be syndetic if $\mathbb{Z}^d$ can be covered by finitely many translates of E.

We have the following corresponding combinatorial result:

Theorem 1.3. *Let $A \subset \mathbb{Z}^d$ with $d^*(A) > \delta > 0$. Then the set of $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$ such that*

$$d^*\left(\bigcap_{\epsilon \in \{0,1\}^d} \{A + (n_1\epsilon_1, \dots, n_d\epsilon_d)\} \right) \ge \delta^{2^d}$$

is syndetic.

1.2. **History of the problem.** In the case where $T_1 = T_2 = \cdots = T_d = T$, the average (1) is

$$\prod_{i=1}^{d} \frac{1}{N_i - M_i} \sum_{\substack{n_i \in [M_i, N_i) \\ i=1,\dots,d}} \prod_{\substack{\epsilon \in \{0,1\}^d \\ \epsilon \neq 00\cdots 0}} T^{n_1\epsilon_1 + \cdots + n_d\epsilon_d} f_\epsilon. \tag{4}$$

The norm convergence of (4) was proved by Bergelson for $d = 2$ in [4], and more generally, by Host and Kra for $d > 2$ in [7]. The related pointwise convergence problem was studied by Assani and he showed that the averages (4) converge a.e. in [1].

The lower bound for the average (3) was firstly studied by Leibman, he provided some lower bounds for (3) in [8]. In the same paper,

he gave an example showing that the average (2) can diverge if the transformations do not commute.

However, Assani showed in [1] that the averages

$$\frac{1}{N^2}\sum_{n,m=1}^{N} f(T_1^n x)g(T_2^m x)h(T_3^{n+m}x)$$

do converge a.e. even if the transformations do not necessarily commute. He extended this result to the case of six functions in [2].

The norm convergence of multiple ergodic averages with several commuting transformations of the form

$$\frac{1}{N}\sum_{n=1}^{N} T_1^n f_1 \cdot \ldots \cdot T_d^n f_d, \tag{5}$$

was proved by Conze and Lesigne [5] when $d = 2$. The general case was originally proved by Tao [6], and subsequent proofs were given by Austin [3], Host [6] and Towsner [10].

1.3. **Methods.** The main tools we use in this paper are the seminorms and the existence of "magic extensions" for commuting transformations established by Host [6]. The "magic extensions" can be viewed as a concrete form of the pleasant extensions built by Austin in [3].

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2. Seminorm and upper bound

2.1. **Notation and definitions.** For an integer $d \geq 1$, we write $[d] = \{1, 2, \ldots, d\}$ and identify $\{0, 1\}^d$ with the family of subsets of $[d]$. Therefore, the assertion "$i \in \epsilon$" is equivalent to $\epsilon_i = 1$. In particular, $\emptyset$ is the same as $00\cdots 0 \in \{0, 1\}^d$. We write $|\epsilon| = \sum_i \epsilon_i$ for the number of elements in ϵ.

Let $(X, \mu, T_1, \cdots, T_d)$ be a system. For each $n = (n_1, \ldots, n_d)$, $\epsilon = \{i_1, \ldots, i_k\} \subset [d]$, and for each integer $1 \leq k \leq d$, we write

$$T_\epsilon^n = T_{i_1}^{n_{i_1}} \cdots T_{i_k}^{n_{i_k}}.$$

For any transformation S of some probability space, we denote by $\mathcal{I}(S)$ the σ-algebra of S-invariant sets.

We define a measure μ_1 on X^2 by

$$\mu_1 = \mu \times_{\mathcal{I}(T_1)} \mu_1.$$

This means that for $f_0, f_1 \in L^\infty(\mu)$, we have

$$\int (f_0 \otimes f_1)(x_0, x_1) d\mu_1(x_0, x_1) = \int \mathbb{E}(f_0|\mathcal{I}(T_1)) \cdot \mathbb{E}(f_1|\mathcal{I}(T_1)) d\mu.$$

For $2 \le k \le d$, we define a measure μ_k (see [6]) on X^{2^k} by

$$\mu_k = \mu_{k-1} \times_{\mathcal{I}(T_k^\triangle)} \mu_{k-1},$$

where $T_k^\triangle := \underbrace{T_k \times \cdots \times T_k}_{2^{k-1}}$.

We write $X^* = X^{2^d}$, and points of X^* are written as $x = (x_\epsilon : \epsilon \subset [d])$. We write $\mu^* := \mu_d$.

For $f \in L^\infty(\mu)$, define

$$|\!|\!| f |\!|\!|_{T_1,\dots,T_d} := \left(\int \prod_{\epsilon \in \{0,1\}^d} f(x_\epsilon) d\mu^*(x) \right)^{1/2^d}.$$

It was shown in Proposition 2 in [6] that $|\!|\!| \cdot |\!|\!|_{T_1,\dots,T_d}$ is a seminorm on $L^\infty(\mu)$. We call this the box seminorm associated to $T_1, \dots, T_d$.

For $\epsilon \subset [d]$, $\epsilon \neq \emptyset$, we write $|\!|\!| \cdot |\!|\!|_\epsilon$ for the seminorm on $L^\infty(\mu)$ associated to the transformations T_i, $i \in \epsilon$. For example, $|\!|\!| \cdot |\!|\!|_{110\cdots 00}$ is the seminorm associated to T_1, T_2 because $\epsilon = 110\cdots 0 \in \{0,1\}^d$ is identified with $\{1,2\} \subset [d]$.

By Proposition 3 in [6], if we rearrange the order of the digits in ϵ, the seminorm $|\!|\!| \cdot |\!|\!|_\epsilon$ remains unchanged.

2.2. **Upper bound.** In the following, we assume that all functions f_ϵ, $\epsilon \subset [d]$, are real valued and satisfy $|f_\epsilon| \le 1$.

Proposition 2.1. Maintaining the above notation and hypotheses,

(6)

$$\limsup_{\substack{N_i - M_i \to \infty \\ i=1,\dots,d}} \Big\| \prod_{i=1}^d \frac{1}{N_i - M_i} \sum_{n \in [M_1,N_1) \times \cdots \times [M_d,N_d)} \prod_{\substack{\epsilon \subset [d] \\ \epsilon \neq \emptyset}} T_\epsilon^n f_\epsilon \Big\|_{L^2(\mu)} \le \min_{\substack{\epsilon \subset [d] \\ \epsilon \neq \emptyset}} |\!|\!| f_\epsilon |\!|\!|_{T_1 \dots T_d}.$$

Proof. We proceed by induction on d. For $d = 1$, we have

$$\Big\| \frac{1}{N_1 - M_1} \sum_{n_1 \in [M_1,N_1)} T_1^{n_1} f_1 \Big\|^2_{L^2(\mu)} \to \int \mathbb{E}(f_1|\mathcal{I}(T_1))^2 d\mu = |\!|\!| f_1 |\!|\!|^2_{T_1}.$$

Let $d \ge 2$ and assume that (6) is established for $d-1$ transformations.

We show that for every $\alpha \subset [d], \alpha \neq \emptyset$, the limsup of the the left hand side of (6) is bounded by $|\!|\!| f_\alpha |\!|\!|_{T_1,\dots,T_d}$. By a permutation of digits

if needed, we can assume that $\alpha \neq \underbrace{0\dots0}_{d-1}1$. The square of the norm in the left hand side of (6) is equal to

$$\Big\|\frac{1}{N_d - M_d}\sum_{n_d\in[M_d,N_d)} T_d^{n_d} f_{0\dots01}\cdot\prod_{i=1}^{d-1}\frac{1}{N_i-M_i}\sum_{m\in[M_1,N_1)\times\cdots\times[M_{d-1},N_{d-1})} \prod_{\substack{\eta\subset[d-1]\\ \eta\neq\emptyset}} T_\eta^m(f_{\eta0}\cdot T_d^{n_d}f_{\eta1})\Big\|^2_{L^2(\mu)}.$$

By the Cauchy-Schwartz Inequality, this is less than or equal to

$$\frac{1}{N_d - M_d}\sum_{n_d\in[M_d,N_d)} \Big\|\prod_{i=1}^{d-1}\frac{1}{N_i-M_i}\sum_{m\in[M_1,N_1)\times\cdots\times[M_{d-1},N_{d-1})} \prod_{\substack{\eta\subset[d-1]\\ \eta\neq\emptyset}} T_\eta^m(f_{\eta0}\cdot T_d^{n_d}f_{\eta1})\Big\|^2_{L^2(\mu)}. \tag{7}$$

By the induction hypothesis, when $N_i - M_i \to \infty$, $i = 1, \dots, d-1$, the limsup of the square of the norm in (7) is less than or equal to

$$\min_{\substack{\eta\subset[d-1]\\ \eta\neq\emptyset}} \|f_{\eta0}\cdot T_d^{n_d} f_{\eta1}\|^2_{T_1,\dots,T_{d-1}},$$

where $\|\cdot\|_{T_1,\dots,T_{d-1}}$ is the seminorm associated to the $d-1$ transformations $T_1,\dots,T_{d-1}$.

Note that α is equal to $\eta 0$ or $\eta 1$ for some $\eta \subset [d-1]$, and by the Cauchy-Schwartz Inequality, we have

$$\begin{aligned}
&\lim_{N_d-M_d\to\infty} \frac{1}{N_d-M_d} \sum_{n_d\in[M_d,N_d)} \|f_{\eta 0}\cdot T_d^{n_d} f_{\eta 1}\|_{T_1,\ldots,T_{d-1}}^{2^{d-1}} \\
&= \lim_{N_d-M_d\to\infty} \frac{1}{N_d-M_d} \sum_{n_d\in[M_d,N_d)} \int \bigotimes_{\eta\subset[d-1]} (f_{\eta 0}\cdot T_d^{n_d} f_{\eta 1}) d\mu_{d-1} \\
&= \int \mathbb{E}(\bigotimes_{\eta\subset[d-1]} f_{\eta 0}|\mathcal{I}(T_d^{\triangle}) \cdot \mathbb{E}(\bigotimes_{\eta\subset[d-1]} f_{\eta 1}|\mathcal{I}(T_d^{\triangle}) d\mu_{d-1} \\
&\leq \left(\int |\mathbb{E}(\bigotimes_{\alpha\subset[d]} f_\alpha|\mathcal{I}(T_d^{\triangle}))|^2 d\mu_{d-1}\right)^{1/2} \\
&= \left(\int \bigotimes_{\alpha\subset[d]} f_\alpha d\mu^*\right)^{1/2} = |||f_\alpha|||_{T_1,\ldots,T_d}^{2^{d-1}}.
\end{aligned}$$

This completes the proof. $\square$

The following proposition is a generalization of Proposition 2.1, although its proof depends upon Proposition 2.1.

Proposition 2.2. *Let r be an integer with $1 \leq r \leq d$. Then*

(8)
$$\limsup_{\substack{N_i-M_i\to\infty\\ i=1,\ldots,d}} \left\| \prod_{i=1}^d \frac{1}{N_i-M_i} \sum_{n\in[M_1,N_1)\times\cdots\times[M_d,N_d)} \prod_{\substack{\epsilon\subset[d]\\ 0<|\epsilon|\leq r}} T_\epsilon^n f_\epsilon \right\|_{L^2(\mu)} \leq \min_{\substack{\epsilon\subset[d]\\ |\epsilon|=r}} |||f_\epsilon|||_\epsilon$$

Proof. We show that for every $\alpha \subset [d]$ with $|\alpha| = r$, the lim sup in (8) is bounded by $|||f_\alpha|||_\alpha$.

By a permutation of digits we can restrict to the case that

$$\alpha = \underbrace{11\ldots1}_{r}00\ldots0\ .$$

We show that

(9)
$$\limsup_{\substack{N_i-M_i\to\infty\\ i=1,\ldots,d}} \left\| \prod_{i=1}^d \frac{1}{N_i-M_i} \sum_{n\in[M_1,N_1)\times\cdots\times[M_d,N_d)} \prod_{\substack{\epsilon\subset[d]\\ 0<|\epsilon|\leq r}} T_\epsilon^n f_\epsilon \right\|_{L^2(\mu)} \leq |||f_\alpha|||_\alpha.$$

The norm in (9) is equal to

$$\| \prod_{i=r+1}^{d} \frac{1}{N_i - M_i} \sum_{m \in [M_{r+1},N_{r+1}) \times \cdots \times [M_d,N_d)} \prod_{\substack{\epsilon \subset \{r+1,\ldots,d\} \\ \epsilon \neq \emptyset}} T_\epsilon^m f_\epsilon \cdot \prod_{j=1}^{r} \frac{1}{N_j - M_j} \sum_{n \in [M_1,N_1) \times \cdots \times [M_r,N_r)} \Big(\prod_{\substack{\eta \subset [r] \\ \eta \neq \emptyset}} T_\eta^n \prod_{\substack{\theta \subset [d-r] \\ |\eta\theta| \le r}} T_{r+\theta}^n f_{\eta\theta} \Big) \cdot (T_1^{n_1} \cdots T_r^{n_r} f_\alpha) \big\|_{L^2(\mu)}. \tag{10}$$

where $r + \theta = \{r + k\colon \ k \in \theta\}$.

Let

$$g_\eta = \begin{cases} \displaystyle\prod_{\substack{\theta \subset [d-r] \\ |\eta\theta| \le r}} T_{r+\theta}^n f_{\eta\theta} & 0 < |\eta| < r \ ; \\ f_\alpha & |\eta| = r \ . \end{cases}$$

Then (10) is equal to

$$\Big\| \prod_{i=r+1}^{d} \frac{1}{N_i - M_i} \cdot \sum_{m \in [M_{r+1},N_{r+1}) \times \cdots \times [M_d,N_d)} \Big(\prod_{\substack{\epsilon \subset \{r+1,\ldots,d\} \\ \epsilon \neq \emptyset}} T_\epsilon^m f_\epsilon \Big) \cdot \prod_{j=1}^{r} \frac{1}{N_j - M_j} \cdot \sum_{n \in [M_1,N_1) \times \cdots \times [M_r,N_r)} \Big(\prod_{\substack{\eta \subset [r] \\ \eta \neq \emptyset}} T_\eta^n g_\eta \Big) \Big\|_{L^2(\mu)}. \tag{11}$$

By the Cauchy-Schwartz Inequality, the square of (11) is less than or equal to

$$\prod_{i=r+1}^{d} \frac{1}{N_i - M_i} \sum_{m \in [M_{r+1},N_{r+1}) \times \cdots \times [M_d,N_d)} \Big\| \prod_{j=1}^{r} \frac{1}{N_j - M_j} \sum_{n \in [M_1,N_1) \times \cdots \times [M_r,N_r)} \prod_{\substack{\eta \subset [r] \\ \eta \neq \emptyset}} T_\eta^n g_\eta \Big\|_{L^2(\mu)}^2. \tag{12}$$

By Proposition 2.1, the limsup of (12) as $N_i - M_i \to \infty$, $i = 1, \ldots, r$ is bounded by

$$\prod_{i=r+1}^{d} \frac{1}{N_i - M_i} \sum_{\substack{n_i \in [M_i,N_i) \\ i=r+1,\ldots,d}} \|f_\alpha\|_{T_1,\ldots,T_r}^2 = \|f_\alpha\|_{T_1,\ldots,T_r}^2.$$

This completes the proof. $\square$

3. The case of the magic extension

We recall the definition of a "magic" system.

Definition 3.1 (Host, [6])**.** A system $(X, \mu, T_1, \dots, T_d)$ is called a "magic" system if $f \in L^\infty(\mu)$ is such that $\mathbb{E}(f | \bigvee_{i=1}^d \mathcal{I}(T_i)) = 0$, then $\| f \|_{T_1,\dots,T_d} = 0$.

Given a system $(X, \mu, T_1, \dots, T_d)$, let X^* and μ^* be defined as in Section 2.1. We denote by T_i^* the *side transformations* of X^*, given by

$$\text{for every } \epsilon \in \{0,1\}^d, \quad (T_i^* x)_\epsilon = \begin{cases} T_i x_\epsilon & \text{if } \epsilon_i = 0 \ ; \\ x_\epsilon & \text{if } \epsilon_i = 1 \ . \end{cases}$$

By Theorem 2 in [6], $(X^*, \mu^*, T_1^*, \dots, T_d^*)$ is a "magic" system, and admits $(X, \mu, T_1, \dots, T_d)$ as a factor.

For $\epsilon \subset [d]$, $\epsilon \neq \emptyset$, we write $\| \cdot \|_\epsilon^*$ for the seminorm on $L^\infty(\mu^*)$ associated to the transformations T_i^*, $i \in \epsilon$. Moreover, we define the σ-algebra

$$\mathcal{Z}_\epsilon^* := \bigvee_{i \in \epsilon} \mathcal{I}(T_i^*)$$

of (X^*, μ^*). For example, $\mathcal{Z}^*_{\{1,2,d\}} = \mathcal{I}(T_1^*) \vee \mathcal{I}(T_2^*) \vee \mathcal{I}(T_d^*)$.

We prove Theorem 1.1 for the magic system $(X^*, \mu^*, T_1^*, \dots, T_d^*)$.

Theorem 3.2. *Let f_ϵ, $\epsilon \subset [d]$, be functions on X^* with $\|f_\epsilon\|_{L^\infty(\mu^*)} \leq 1$ for every ϵ. Then the averages*

$$\prod_{i=1}^d \frac{1}{N_i - M_i} \sum_{n \in [M_1,N_1) \times \cdots \times [M_d,N_d)} \prod_{\substack{\epsilon \subset [d] \\ \epsilon \neq \emptyset}} T_\epsilon^{*n} f_\epsilon \tag{13}$$

converge in $L^2(\mu^*)$ for all sequences of intervals $[M_1, N_1), \dots, [M_d, N_d)$ whose lengths $N_i - M_i$ $(1 \leq i \leq d)$ tend to ∞.

Since the system $(X^*, \mu^*, T_1^*, \dots, T_d^*)$ admits $(X, \mu, T_1, \dots, T_d)$ as a factor, Theorem 3.2 implies our main result Theorem 1.1.

Theorem 3.3. *For every $\epsilon \subset [d]$, $\epsilon \neq \emptyset$, and every function $f \in L^\infty(\mu^*)$, we have:*

$$\textit{If } \mathbb{E}_{\mu^*}(f \mid \mathcal{Z}_\epsilon^*) = 0, \ \textit{then } \| f \|_\epsilon^* = 0. \tag{14}$$

Proof. Assume $|\epsilon| = r > 0$. By a permutation of digits we can assume that

$$\epsilon = \{d - r + 1, d - r + 2, \dots, d\}.$$

We define a new system $(Y, \nu, S_1, \dots, S_r)$, where $Y = X^{2^{d-r}}$, $\nu = \mu_{d-r}$, the $d-r$ step measure associated to $T_1^*, \dots, T_{d-r}^*$. Define

$$S_i = \underbrace{T_{d-r+i} \times \cdots \times T_{d-r+i}}_{2^{d-r}}$$

on Y for $i = 1, \dots, r$.

Note that by definition, $Y^* = X^{2^d} = X^*$, and

$$S_i^* = T_{d-r+i}^*, \quad S_i^{\triangle} = T_{d-r+i}^{\triangle}$$

for $i = 1, \dots, r$. Moreover,

$$\nu_1 = \nu \times_{\mathcal{I}(S_1)} \nu = \mu_{d-r} \times_{\mathcal{I}(T_{d-r+1}^{\triangle})} \mu_{d-r} = \mu_{d-r+1}.$$

By induction,

$$\nu_{i+1} = \nu_i \times_{\mathcal{I}(S_{i+1}^{\triangle})} \nu_i = \mu_{d-r+i} \times_{\mathcal{I}(T_{d-r+i+1}^{\triangle})} \mu_{d-r+i} = \mu_{d-r+i+1},$$

for $i = 1, \dots, r-1$.

Therefore $(X^*, \mu^*, T_{d-r+1}^*, \dots, T_d^*)$ is just the magic extension $(Y^*, \nu^*, S_1^*, \dots, S_r^*)$ of $(Y, \nu, S_1, \dots, S_r)$. So

$$\mathcal{Z}_\epsilon^* = \bigvee_{i\in\epsilon} \mathcal{I}(T_i^*) = \bigvee_{i=1}^{r} \mathcal{I}(S_i^*) := \mathcal{W}_Y^*.$$

If $f \in L^\infty(\mu^*)$ with $\mathbb{E}_{\mu^*}(f \mid \mathcal{Z}_\epsilon^*) = 0$, this is equivalent to $\mathbb{E}_{\mu^*}(f \mid \mathcal{W}_Y^*) = 0$, and by Theorem 2 in [6], we have $\|\!|f|\!\|_{S_1^*,\dots,S_r^*}^* = 0$. Thus $\|\!|f|\!\|_\epsilon^* = \|\!|f|\!\|_{S_1^*,\dots,S_r^*}^* = 0$ □

Proposition 3.4. Let f_ϵ, $\epsilon \subset [d]$, be functions on X^* with $\|f_\epsilon\|_{L^\infty(\mu^*)} \leq 1$ for every ϵ. Let r be an integer with $1 \leq r \leq d$. Then the averages

$$\prod_{i=1}^{d} \frac{1}{N_i - M_i} \sum_{n\in[M_1,N_1)\times\cdots\times[M_d,N_d)} \prod_{\substack{\epsilon\subset[d] \\ 0<|\epsilon|\leq r}} T_\epsilon^{*n} f_\epsilon \tag{15}$$

converge in $L^2(\mu^*)$ for all sequences of intervals $[M_1, N_1), \dots, [M_d, N_d)$ whose lengths $N_i - M_i$ $(1 \leq i \leq d)$ tend to ∞.

We remark that Theorem 3.2 follows immediately from this proposition when $r = d$.

Proof. We proceed by induction on r. When $r = 1$, the average (15) is

$$\prod_{i=1}^{d} \frac{1}{N_i - M_i} \sum_{\substack{n_i\in[M_i,N_i) \\ i=1,\dots,d}} T_1^{*n_1} f_{10\dots0} \cdots T_d^{*n_d} f_{0\dots01}. \tag{16}$$

By the Ergodic Theorem, this converges to $\mathbb{E}(f_{10\dots0}|\mathcal{I}(T_1^*)) \cdots \mathbb{E}(f_{0\cdots01}|\mathcal{I}(T_d^*))$.

Assume $r > 1$, and that the proposition is true for $r-1$ transformations.

For $\alpha \subset [d]$, $|\alpha| = r$, if $\mathbb{E}_{\mu^*}(f_\alpha \mid \mathcal{Z}^*_\alpha) = 0$, then by Theorem 3.3, we have $|||f_\alpha|||^*_\alpha = 0$. By Proposition 2.1, the average (15) converges to 0. Otherwise, by a density argument, we can assume that

$$f_\alpha = \prod_{i\in\alpha} f_{\alpha,i}$$

where $f_{\alpha,i}$ is T^*_i-invariant. Then

$$T^{*n}_\alpha f_\alpha = \prod_{i\in\alpha} T^{*n}_{\alpha\setminus\{i\}} f_{\alpha,i} \ .$$

Thus

$$\prod_{\substack{\epsilon\subset[d]\\ 0<|\epsilon|\le r}} T^{*n}_\epsilon f_\epsilon = \prod_{\substack{\eta\subset[d]\\ 0<|\eta|\le r-1}} T^{*n}_\eta g_\eta \ ,$$

where

$$g_\eta = \begin{cases} f_\eta & |\eta| < r-1 \ ; \\ f_\eta \prod_{i\notin\eta} f_{\eta\cup i,i} & |\eta| = r-1 \ . \end{cases}$$

Therefore (15) converges by the induction hypothesis. □

4. Combinatorial interpretation

Proof of Theorem 1.2. Apply Theorem 1.1 to the indicator function $\mathbf{1}_A$, we know that the limit of the averages

$$\prod_{i=1}^{d} \frac{1}{N_i - M_i} \sum_{\substack{n_i\in[M_i,N_i)\\ i=1,\dots,d}} \int \prod_{\epsilon\in\{0,1\}^d} T_1^{n_1\epsilon_1}\cdots T_d^{n_d\epsilon_d}\mathbf{1}_A d\mu \tag{17}$$

exist. By Lemma 1 in [6], if we take the limit as $N_1 - M_1 \to \infty$, then as $N_2 - M_2 \to \infty$, ... and then as $N_d - M_d \to \infty$, the average (17) converges to $|||\mathbf{1}_A|||^{2^d}_{T_1,\dots,T_d}$. Thus the limit of the average (17) is $|||\mathbf{1}_A|||^{2^d}_{T_1,\dots,T_d}$. Since

$$|||f|||^{2^d}_{T_1,\dots,T_d} = \|\mathbb{E}(\bigotimes_{\epsilon\subset[d-1]} f|\mathcal{I}(T_d^{\triangle}))\|^2_{L^2(\mu_{d-1})} \ge (\int \bigotimes_{\epsilon\subset[d-1]} f d\mu_{d-1})^2 = |||f|||^{2^d}_{T_1,\dots,T_{d-1}},$$

we have $|||\mathbf{1}_A|||_{T_1,\dots,T_d} \ge |||\mathbf{1}_A|||_{T_1} \ge \int \mathbf{1}_A d\mu = \mu(A)$, and the result follows. □

Theorem 1.2 has the following corollary:

Corollary 4.1. Let $(X, \mathcal{B}, \mu, T_1, \ldots, T_d)$ be a system, where $T_1, \ldots, T_d$ are commuting measure preserving transformations, and let $A \in \mathcal{B}$. Then for any $c > 0$, the set of $n \in \mathbb{Z}^k$ such that

$$\mu\Big(\bigcap_{\epsilon \in \{0,1\}^d} T_1^{-n_1\epsilon_1} \cdots T_d^{-n_d\epsilon_d} A \Big) \geq \mu(A)^{2^d} - c$$

is syndetic.

The proof is exactly the same as Corollary 13.8 in [7].

Theorem 1.3 follows by combining Furstenberg's correspondence principle and Corollary 4.1.

References

1. I. Assani. *Pointwise convergence of ergodic averages along cubes.* To appear, J. Anal. Math.
2. I. Assani. *Averages along cubes for not necessarily commuting measure preserving transformations.* Contemp. Math. vol. **430**, 1-19, 2007.
3. T. Austin. *On the norm convergence of nonconventional ergodic averages.* To appear, Ergodic Theory Dynam. Systems.
4. V. Bergelson. *The multifarious Poincaré recurrence theorem. Descriptive set theory and dynamical systems* (Marseille-Luminy, 1996) (M. Foreman, A. S. Kechris, A. Louveau, and B. Weiss, eds.), London Math. Soc. Lecture Note Ser. **277** Cambridge Univ. Press, Cambridge (2000), 31-57.
5. J. P. Conze and E. Lesigne. *Théorèmes ergodiques pour des mesures diagonales.* Bull. Soc. Math. France **112** (1984), no. 2, 143–175.
6. B. Host. *Ergodic seminorms for commuting transformations and applications.* Studia Math. **195** (2009), no. 1, 31-49.
7. B. Host and B. Kra, *Nonconventional ergodic averages and nilmanifolds.* Ann. of Math. **161** (2005), no. 1, 397-488.
8. A. Leibman. *Lower bounds for ergodic averages.* Ergodic Theory Dynam. Systems **22** (2002), 863-872
9. T. Tao. *Norm convergence of multiple ergodic averages for commuting transformations.* Ergodic Theory Dynam. Systems **28** (2008), 657-688.
10. H. Towsner. *Convergence of diagonal ergodic averages.* Ergodic Theory Dynam. Systems **29** (2009), no. 4, 1309-1326.

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